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Generalised Strip Yield Crack Arrest Model for a Piezoelectric Strip with Transverse Crack

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Abstract : A study is carried for arrest of opening of a crack and fatigue crack growth rate for a piezoelectric ceramic strip weakened by a transverse, finite, hairline straight crack. A strip yield model is considered under anti-plane shear stress and in-plane electric loading conditions. The developed slide zone rims are subjected to quadratically varying yield point cohesive anti-plane shear stress to arrest the crack from further opening. Fourier integral transform method is employed, which reduces problem to the solution of Fredholm integral equation of second kind. This integral equation in turn is solved numerically. Expressions are derived for the length of slide zone, crack sliding displacement and crack growth rate. A qualitative study is presented for the parameters affecting the opening of the crack with respect to the strip width, material constants etc. in the form of the graphs. The results obtained are analyzed and conclusions are drawn.

Key words : Crack growth rate, crack sliding displacement, piezoelectric ceramic strip, slide zone, stress intensity factors.

Introduction

Ever search for light weight, strong, durable and smart materials for technological purposes have drawn attention towards the poled piezoelectric materials. By now the piezoelectric crystals have proven their wide utility in medical technology, submarine technology, electronic equipments etc. Its wide utility has attracted researchers to investigate the behavior of mechanics of cracking for piezoelectric ceramics. A lot of work has been reported on cracks in infinite piezoelectric ceramics *viz.* Pak (1992), Dunn (1994), Park and Sun (1995), Zhang *et al*. (1996), Bhargava and Saxena (2005) to quote few.

As the present paper deals with cracked piezoelectric strip problem, an attempt is made to discuss the development of research on such types of problems below:

Shindo *et al*. (1990) found singular stress and electric fields for a cracked piezoelectric strip. These studies are further extended to investigate the electroelastic intensification near the anti-plane shear crack in an orthotropic piezoelectric ceramic strip using theory of linear piezoelectricity by Shindo *et al*. (1996). Narita and Shindo (1998) proposed a generalized Dugdale model for cracked piezoelectric ceramic strip to study the fatigue crack growth rate under anti-plane shear stress and in-plane electrical loading conditions. Yu and Chen (1998) studied the transient response of a cracked infinite piezoelectric strip under anti-plane

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impact. Wang and Noda (2000) obtained the solution of the generalized plane deformation problem of a piezoelectric material strip with crack using Laplace and Fourier transforms method. Wang and Mai (2002) dealt with the problem of a cracked piezoelectric material strip under combined mechanical and electrical loads. Both permeable crack and impermeable crack assumptions are considered by them. Wang (2004) considered a mode-III crack problem for functionally graded piezoelectric material strip, for the case when mechanical and electrical properties of the strip are considered to be of the class of functions for which the equilibrium equations have an analytic solution. Zhang and Deng (2005) formulated cohesive zone model for studying crack initiation and crack propagation phenomena. The method of determination of the pre-fracture zone length and the crack opening displacement for a plane-strain problem of electrically permeable crack located in a thin interlayer between two identical piezoelectric materials is suggested by Loboda *et al*. (2006). A finite mode-III crack in a piezoelectric semiconductor of 6 mm crystals is analyzed by Hu *et al*. (2007) using Fourier transform method.

A generalized Dugdale crack arrest model solution is obtained in this paper for a piezoelectric strip weakened by a transverse straight crack, the developed slide zones are arrested by anti-plane quadratically varying yield point shear stress. Expressions are derived for crack sliding zone and crack sliding displacement. Crack growth rate is also obtained under cyclic loading.

Mathematical Formulation

A poled piezoelectric strip occupies *oxyz* plane. The strip is poled along oz direction and is thick enough in *z*- direction to allow the state of anti-plane shear (Fig. 1). In this case the boundary value problem is simplified considerably if one considers only the out-of-plane displacement u_i and in-plane electric field E_i ($i = x, y, z$) such that

$$u_x = u_y = 0 \text{ and } u_z = w(x, y), \qquad \text{...... (1)}$$

$$E_x = E_x(x,y),\ E_y = E_y(x,y) \text{ and } E_z = 0. \qquad \text{...... (2)}$$

The constitutive relations for transversely isotropic piezoelectric ceramics may be written as

$$\sigma_{xz} = c_{44} w_{,x} - e_{15} E_x \ , \qquad \text{...... (3)}$$

$$\sigma_{yz} = c_{44} w_{,y} - e_{15} E_y \ , \qquad \text{...... (4)}$$

$$D_x = e_{15} w_{,x} + \varepsilon_{11} E_x \ , \qquad \text{...... (5)}$$

$$D_y = e_{15} w_{,y} + \varepsilon_{11} E_y \ , \qquad \text{...... (6)}$$

where σ_{xz} and σ_{yz} are the shear stresses, D_x and D_y are the electric displacement components in x, y directions. The elastic stiffness constant c_{44} is measured in a constant electric field, the dielectric constant ε_{11} is measured at constant strain and piezoelectric constant is denoted by e_{15}. A comma implies the partial differentiation with respect to the argument following it.

As is well-known the electric field components are related to electric potential ϕ by

$$E_x = -\phi_{,x} \quad \text{and} \quad E_y = -\phi_{,y} \qquad \text{...... (7)}$$

Constitutive equations for this case are obtained by substituting values from governing equations (3-6) into stress and electric displacement equilibrium equations,

.

These may be written as

$$c_{44}\nabla^2 w + e_{15}\nabla^2\phi = 0, \qquad \text{...... (8)}$$

$$e_{15}\nabla^2 w - \varepsilon_{11}\nabla^2\phi = 0. \qquad \text{...... (9)}$$

Note that $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ denotes the two- dimensional Laplacian operator.

Solution for equations (8, 9) may be written using Fourier integral transform method and taking the inverse Fourier transform, as

$$w(x,y) = \frac{2}{\pi}\int_0^\infty \left\{A_1(\alpha)e^{-\alpha y}\cos(\alpha x) + A_2(\alpha)\cosh(\alpha x)\sin(\alpha y)\right\}d\alpha + a_\infty y \qquad (10)$$

$$\phi(x,y) = \frac{2}{\pi}\int_0^\infty \left\{B_1(\alpha)e^{-\alpha y}\cos(\alpha x) + B_2(\alpha)\cosh(\alpha x)\sin(\alpha y)\right\}d\alpha - b_\infty y \qquad (11)$$

where $A_i(\alpha)$ and $B_i(\alpha)$ ($i=1, 2$) are the unknown functions to be determined a_∞ and b_∞, are the arbitrary constants determined from the boundary conditions at infinity.

The Problem and Solution

A piezoelectric strip occupies the region $-h \le x \le h$, in $oxyz$ coordinate system (Fig.1) and is thick enough in *z*-direction to allow the state of anti-plane shear. The piezoelectric ceramic strip exhibits symmetry of a hexagonal crystal of class 6 mm with respect to principal *x*, *y* and *z* axes. The strip is weakened by a finite, hairline straight crack, L, the crack lies in the interval . The edges of the strip are stress and charge free. The infinite boundary is subjected to anti-plane shear stress and in-plane electric load D_∞ or E_∞ as the case may be. Consequently, the crack yields both mechanically and electrically. The ceramic of the strip being mechanically more brittle a mechanical singularity is encountered first. Consequently, under small-scale yielding, a slide zone protrudes ahead of each tip of the crack. The slide zones Γ_1 and Γ_2 developed at the tips - *a* and *a* occupy the intervals $-b \le x \le -a$ and ; *y=0*, respectively. The rims of the slide zones are subjected to an anti-plane shear stress to arrest the crack from further opening, where τ_{ye} is the yield point shear stress of the strip and x is any point on the slide zone. Because of symmetry in geometry and load conditions, it is sufficient to consider the problem for the region $0 \le x \le h$, . Fig. 1 below depicts the schematic presentation of the configuration of the problem.

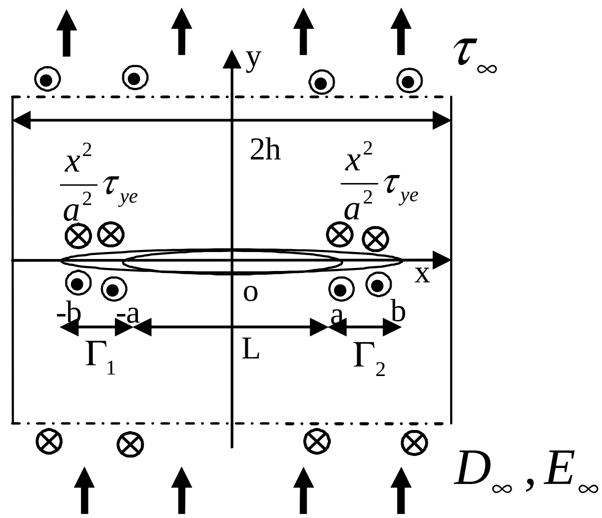

Fig. 1 : Schematic presentation of configuration of the problem

The mathematical model of the above case may be written as follows: The piezoelectric strip occupies the region , is cut along an internal crack ,

y=0 {formed of the union of intervals }. The boundary conditions of the problem mathematically be written as

$$(i)\ \sigma_{yz}(x,0) = \frac{x^2}{a^2}\tau_{ye}H(x-a), \qquad 0 \le x \le b$$

$$(ii)\ w(x,0) = 0, \qquad b \le x \le h$$

$$(iii) E_x(x,0) = E_x^{V}(x,0) \qquad 0 \le x \le a$$

$$(iv)\ \phi(x,0) = 0, \qquad a \le x \le h$$

$$(v)\ D_y(x,0) = D_y^{V}(x,0) \qquad 0 \le x \le a$$

$$(vi) D_x(h,y) = 0, \qquad \textit{for all } y$$

$$(vii)\ \sigma_{xz}(h,y) = 0, \qquad \textit{for all } y.$$

where H( ) denotes Heaviside function. Superscript V denotes that the quantities refer to vaccum inside the crack.

At infinite boundary, two types of boundary conditions are prescribed:

For each case, the desired potentials $w(x,y)$ and $\phi(x,y)$ are obtained using equations (10, 11) determining $A_i(\alpha)$, $B_i(\alpha)$ (i=*1, 2*) and a_∞, b_∞ using the prescribed conditions of the problem.

Solution for Case I: The constants a_∞, b_∞ are determined employing the boundary condition (viii) using (10, 11) together with equations (3-6) as

$$a_\infty^I = \frac{\varepsilon_{11}\tau_\infty + e_{15}D_\infty}{c_{44}\varepsilon_{11} + e_{15}^2} \qquad \text{and} \qquad b_\infty^I = \frac{c_{44}D_\infty - e_{15}\tau_\infty}{c_{44}\varepsilon_{11} + e_{15}^2} \tag{12}$$

Superscript *I* denote that the quantities refer to the Case *I*.

Following two sets of dual integral equations are obtained for determining $A_i(\alpha)$ and $B_i(\alpha)$ (i=*1, 2*) using boundary conditions (i to iv) together with equations (10, 11)

$$\frac{2}{\pi}\int_0^\infty \alpha\left[\left\{c_{44}A_1(\alpha)+e_{15}B_1(\alpha)\right\}\cos(\alpha x)-\left\{c_{44}A_2(\alpha)+e_{15}B_2(\alpha)\right\}\cosh(\alpha x)\right]d\alpha = \tau_\infty - \frac{x^2}{a^2}\tau_{ye}H(x-a) \quad , \quad 0 \le x \le b \tag{13}$$

(*viii*) Case I

$\sigma_{yz}(x,y)=\tau_\infty$, and $D_y(x,y)=D_\infty$ for $0 \le x \le h,\ y \to \infty$

(*ix*) Case II

$\sigma_{yz}(x,y)=\tau_\infty$, and $E_y(x,y)=E_\infty$ for $0 \le x \le h,\ y \to \infty$

$$\int_0^\infty A_1(\alpha)\cos(\alpha x)\,d\alpha = 0 \quad , \quad b \le x \le h \tag{14}$$

$$\int_0^\infty B_1(\alpha)\cos(\alpha x)\,d\alpha = 0 \quad , \quad a \le x \le h \tag{15}$$

$$\int_0^\infty \alpha B_1(\alpha)\sin(\alpha x)\,d\alpha = 0 \quad , \quad 0 \le x \le a \tag{16}$$

Two new functions $\varphi_1(\xi)$ and $\varphi_2(\xi)$ are introduced to determine $A_1(\alpha)$ and $B_1(\alpha)$ as

$$A_1(\alpha) = \frac{\pi b^2}{2c_{44}} \int_0^1 \sqrt{\xi}\, \varphi_1(\xi) J_0(b\alpha\xi) d\xi \qquad (17)$$

$$B_1(\alpha) = \frac{\pi a^2}{2c_{44}} \int_0^1 \sqrt{\xi}\, \varphi_2(\xi) J_0(a\alpha\xi) d\xi \qquad (18)$$

Equation (18) together with equation (16) yields $B_1(\alpha) = 0$. Unknown functions $A_2(\alpha)$ and $B_2(\alpha)$ are determined using edge boundary conditions (vi and vii) and $B_1(\alpha) = 0$.

These may finally be given as

$$A_2(\alpha) = \frac{2}{\pi\alpha \sinh(\alpha h)} \int_0^\infty \frac{s\alpha}{s^2 + \alpha^2} A_1(\alpha) \sin(sh) ds, \qquad (19)$$

$$B_2(\alpha) = 0. \qquad (20)$$

Boundary conditions (vi and vii) and equations (13, 17) yield following Fredholm's integral equation of the second kind to determine $\varphi_1(\xi)$:

$$\varphi_1(\xi) + \int_0^1 \varphi_1(\eta) k(\xi,\eta) d\eta = \begin{cases} \tau_\infty \sqrt{\xi} & , \quad \xi < \dfrac{a}{b} \\ \tau_\infty \sqrt{\xi} + \dfrac{b\sqrt{\xi}}{\pi a} \tau_{ye} \left[\sqrt{\xi^2 - \dfrac{a^2}{b^2}} + \dfrac{b}{a} \xi^2 \cos^{-1}\left(\dfrac{a}{b\xi} \right) \right] & , \dfrac{a}{b} < \xi < 1 \end{cases} \qquad (21)$$

where kernel $k(\xi,\eta)$ is

$$k(\xi,\eta) = -\sqrt{\xi\eta} \int_0^\infty \frac{\alpha e^{-\alpha h/a}}{\sinh(\alpha h/a)} I_0(\alpha\eta) I_0(\alpha\xi) d\alpha \qquad (22)$$

Note that $I_0()$ denotes zero order modified Bessel's function of the first kind.

Consequently $A_1(\alpha)$ is also determined.

Solution for Case II : The arbitrary constants a_∞, b_∞ for this case are determined using boundary condition (ix) and may be written as-

$$a_\infty^{II} = \frac{\tau_\infty + e_{15} E_\infty}{c_{44}} \quad \text{and} \quad b_\infty^{II} = E_\infty \qquad (23)$$

Superscript II denotes that the quantities refer to Case II. The unknown functions $A_i(\alpha)$ and $B_i(\alpha)$ ($i = 1, 2$) remain same as in Case *I*.

Applications

Stress intensity factor, slide zone and crack sliding displacement

Sliding mode stress intensity factor, $K_{III}(b)$, at the tip $x = b$ is obtained using the expression

$$K_{III}(b) = \lim_{x\to b^+}\{\sqrt{2\pi(x-b)}\,\sigma_{yz}(x,0)\} \tag{24}$$

Sliding mode stress intensity factor for both the Cases I and II, at the tip $x = b$ is given by

$$K_{III}(b) = \sqrt{\pi b}\,\varphi_1(1). \tag{25}$$

The condition that the crack stops propagating at $x = b$ yields a transcendental equation to determine the slide zone length from

$$b^2\cos^{-1}\left(\frac{a}{b}\right) + a\sqrt{b^2 - a^2} = \pi a^2\left(\frac{\tau_\infty - S(h/a)}{\tau_{ye}}\right), \tag{26}$$

where $S(h/a) = \int_0^1 \varphi_1(\eta)k(\xi,\eta)d\eta.$ (27)

Relative crack sliding displacement, $CSD_{III}(x)$, for the rims of the crack is given by

$$CSD_{III}(x) = \frac{4\tau_{ye}}{\pi c_{44}a^2}\left[\left\{b^2\sqrt{b^2-x^2}\,Cos^{-1}\left(\frac{a}{b}\right) + a^2\sqrt{\left(\frac{b}{a}\right)^2 - 1}\right\}\right.$$
$$\left. - \left\{\int_x^a \frac{\alpha^3}{\sqrt{\alpha^2 - x^2}}Cos^{-1}\left(\frac{a}{\alpha}\right)d\alpha + a\int_x^a \alpha\sqrt{\frac{\alpha^2 - c^2}{\alpha^2 - x^2}}d\alpha\right\}\right]. \tag{28}$$

Fatigue crack growth rate

Assuming that the results of static loading conditions can be applied to cyclic loading conditions by making the substitution

$$\tau_\infty \to \frac{\Delta\tau}{2}, \quad \tau_0 \to \frac{\Delta\tau_0}{2}, \quad \tau_{ye} \to \tau_{yc} \quad and \quad \gamma = D_c\tau_{yc}$$

where $\Delta\tau = \tau_2 - \tau_1$; τ_1 and τ_2 represent minimum and maximum applied shear stress at zero electrical loads; τ_{yc} is the cyclic yield strength; D_c denotes the critical value of accumulated plastic displacement.

The fatigue crack growth rate, $\frac{da}{dN}$, under small-scale yielding may be written as-

$$\frac{da}{dN} = \frac{\pi}{192\, c_{44}\, \gamma\, \tau_{yc}^2} \left(\Delta K_{III}\right)^4 \tag{29}$$

where

$$\Delta K_{III} = \left\{\Delta\tau - 2S\left(h/a\right)\right\}\sqrt{\pi a}\,. \tag{30}$$

Case Study

Case study has been presented for crack opening and fatigue crack growth rate for piezoelectric ceramics, PZT-4 and PZT-5H. The material properties of the ceramics are presented below in Table 1.

Table1: Material constants

Material constants	Ceramics	
	PZT-4	PZT-5H
c_{44} (10^{10}N/m^2)	2.56	2.3
e_{15} (C/m^2)	12.7	17
ε_{11} (10^{-10}C/Vm)	64.6	150.4

The material constants are taken from Narita *et al*. (2001).

Fig. 2 depicts the variation of crack growth rate as the width of strip is increased for Case *I*. It may be observed that as the strip width increases, the crack growth rate decreases. Also, it may be noted that as the value of the ratio $(c_{44}e_{15}D_0)/(\tau_0 \overline{c_{44}}\varepsilon_{11})$ is increased from , crack growth rate is considerably reduced.

Similar variation of crack growth rate is plotted for case *II* in Fig.3. In this case also, the crack growth rate reduces and stabilizes as the width of the strip is increased. The crack growth rate becomes independent of ceramic properties. As the ratio $(e_{15}E_0)/\tau_0$ increases from -0.25 to 0.5, the crack growth rate reduces further.

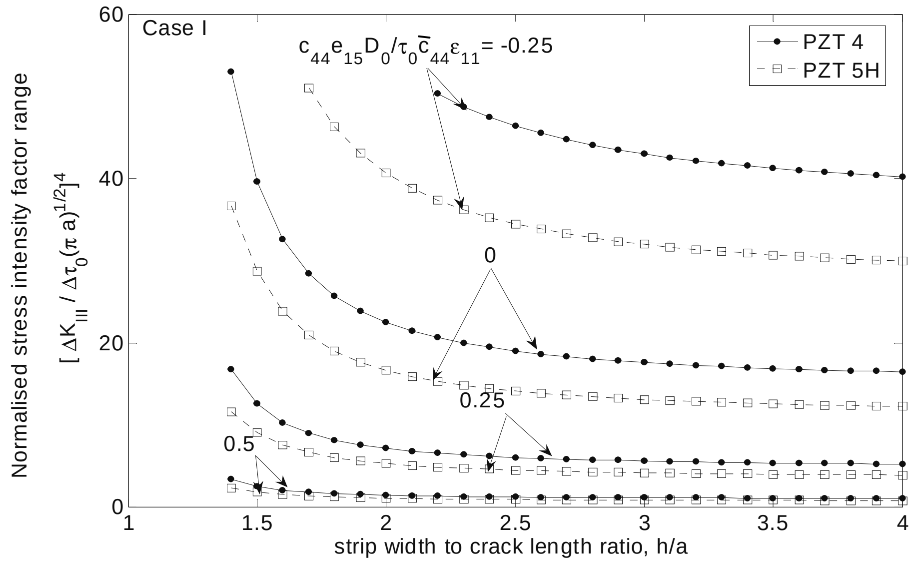

Fig. 2 : Variation of crack growth rate versus strip width to crack length ratio, for Case *I*

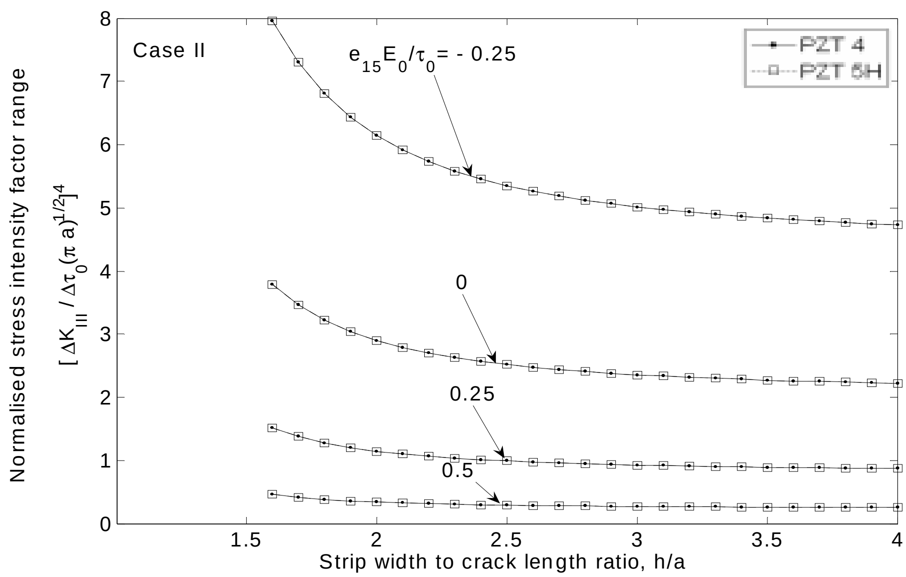

Fig. 3 : Variation of crack growth rate versus strip width to crack length ratio, for Case *II*

Figures 4 & 5 show the variation of crack growth rate as slide zone to crack length ratio, $(b-a)/a$, is increased for Cases *I* and *II* respectively. It can be seen that crack growth rate is almost ten times higher for Case *I* as compare to that for Case *II*. The crack growth rate decreases in both the cases for higher values of the ratio, $(b-a)/a$. For Case *II*, the crack growth rate is independent of ceramic properties. However for Case *I*, it is dependent on ceramic properties. As the ratio is increased from , the crack growth for Case *I* reduces and the variation for different ceramics narrows down and for the value of this ratio equals to 0.5, the variation for both the ceramics overlaps and becomes uniformly constant. In Fig. 5, different curves are drawn for the ratio $(e_{15}E_0)/\tau_0$ varying from. For this case also, as the ratio $(e_{15}E_0)/\tau_0$ assumes positive higher values, the crack growth reduces and becomes uniformly constant.

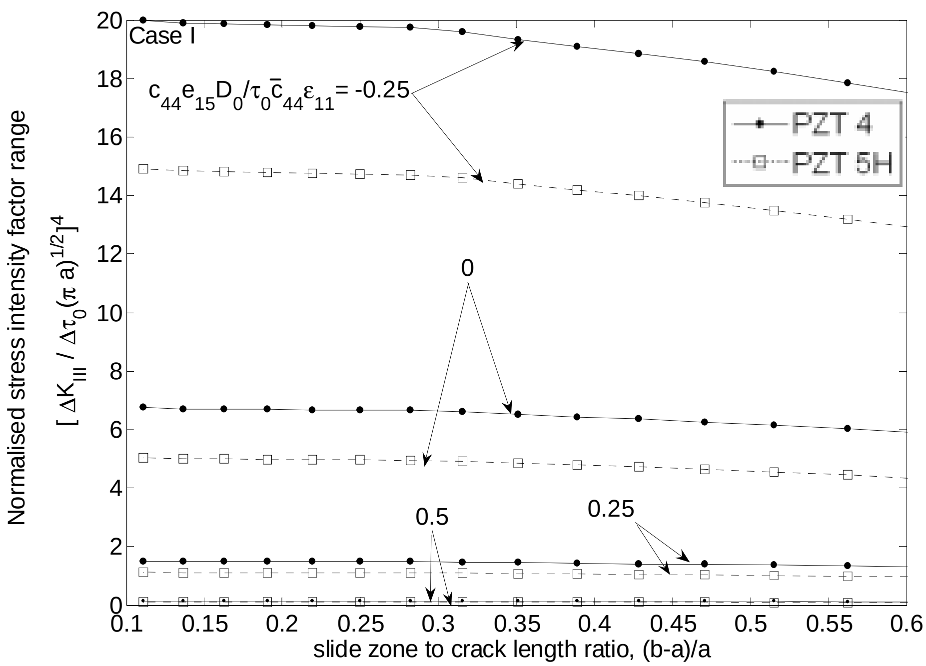

Fig. 4 : Variation of crack growth rate for different values of slide zone to crack length ratio, for Case *I*

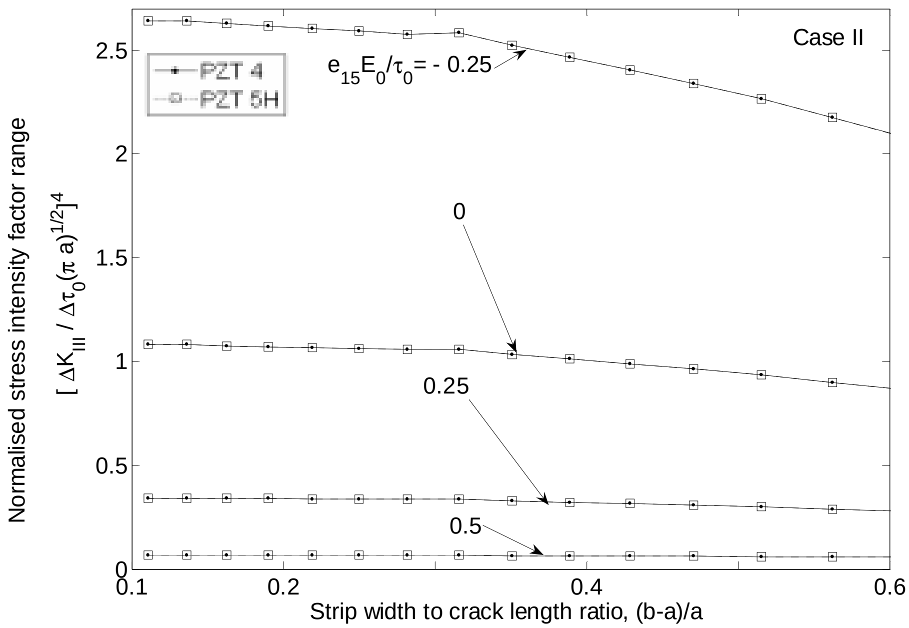

Fig. 5 : Variation of crack growth rate for different values of slide zone to crack length ratio, for Case *II*

Conclusion

Using linear theory of piezoelectricity, a strip yield model is proposed in this paper to arrest the growth of a semi-stationary crack.

Applicability of the model is extended to study the crack growth under cyclic loading. The variation of crack growth rate with respect to width of strip and slide zone length is studied. The results conclude that the crack opening is arrested and the crack growth rate reduces if the proposed model is employed.

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